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# CERTAIN SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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**Abstract.** Let  $\Sigma_p$  be the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \cdots + a_{k+p-1}z^k + \cdots \quad (p \in N = \{1, 2, \dots\})$$

which are analytic in the punctured open unit disk. In this paper, a new subclass  $\Sigma_{n,p}(\alpha, \delta, \mu, \lambda)$  of  $\Sigma_p$  is introduced. For functions  $f \in \Sigma_{n,p}(\alpha, \delta, \mu, \lambda)$ , we find a sufficient condition on the function  $g \in \Sigma_p$  which can guarantee that

$$\operatorname{Re} \left\{ \frac{D^{n+p}f(z)}{D^{n+p}g(z)} \right\} > \alpha \quad (0 \leq \alpha < 1, z \in E = \{z: |z| < 1\})$$

implies

$$\operatorname{Re} \left\{ \frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} \right\} > \beta \quad (0 \leq \alpha < \beta < 1, z \in E).$$

Further, some applications of this result are given.

## 1. Introduction and Preliminaries

Let  $\Sigma_p$  denote the class of functions of the form

$$(1.1) \quad f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \cdots + a_{k+p-1}z^k + \cdots \quad (p \in N = \{1, 2, \dots\})$$

which are analytic in the annulus  $D = \{z: 0 < |z| < 1\}$ . The Hadamard product or convolution of two functions  $f$  and  $g$  in  $\Sigma_p$  will be denoted by  $f * g$ . Following Uralegaddi and Path [6], we define

$$(1.2) \quad \begin{aligned} D^{n+p-1}f(z) &= \frac{1}{z^p(1-z)^{n+p}} * f(z) \\ &= \frac{1}{z^p} \left( \frac{z^{n+2p-1}f(z)}{(n+p-1)!} \right)^{(n+p-1)}, \end{aligned}$$

where  $n$  is any integer greater than  $-p$ . The symbol  $D^{n+p-1}$  when  $p = 1$  was introduced by Uralegaddi and Ganigi [5]. It follows from (1.2) that

$$(1.3) \quad z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z).$$

We denote by  $\sum_{n,p}(\alpha, \delta, \mu, \lambda)$  the class of all functions  $f \in \sum_p$  such that

$$\operatorname{Re}\left\{(1-\lambda)\left(\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right)^\mu + \lambda \frac{D^{n+p}f(z)}{D^{n+p}g(z)}\left(\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right)^{\mu-1}\right\} > \alpha,$$

where  $g \in \sum_p$  satisfies the condition

$$\operatorname{Re}\left\{\frac{D^{n+p-1}g(z)}{D^{n+p}g(z)}\right\} > \delta \quad (0 \leq \delta < 1, z \in E = \{z : |z| < 1\}),$$

where  $\alpha$  and  $\mu$  are real numbers such that  $0 \leq \alpha < 1, \mu > 0$ ,  $n$  is any integer greater than  $-p$  and  $\lambda$  is a complex number such that  $\operatorname{Re}\{\lambda\} > 0$ .

The object of the present paper is to investigate some interesting properties and applications for the class  $\sum_{n,p}(\alpha, \delta, \mu, \lambda)$ .

To establish our main results we need the following lemmas.

LEMMA 1. Let  $\Omega$  be a set in the complex plane  $\mathbb{C}$  and let the function  $\psi : \mathbb{C}^2 \rightarrow \mathbb{C}$  satisfy the condition  $\psi(ir_2, s_1) \notin \Omega$ , for all real  $r_2, s_1 \leq -\frac{1+r_2^2}{2}$ . If  $q(z)$  is analytic in  $E$  with  $q(0) = 1$  and  $\psi(q(z), zq'(z)) \in \Omega, z \in E$ , then  $\operatorname{Re}\{q(z)\} > 0$  in  $E = \{z : |z| < 1\}$ .

A more general form of this lemma may be found in [3].

LEMMA 2([4]). If  $q(z)$  is analytic in  $E$  with  $q(0) = 1$ , and if  $\lambda$  is a complex number satisfying  $\operatorname{Re}\{\lambda\} \geq 0$  ( $\lambda \neq 0$ ), then  $\operatorname{Re}\{q(z) + \lambda zq'(z)\} > \alpha$  ( $0 \leq \alpha < 1$ ) implies  $\operatorname{Re}\{q(z)\} > \alpha + (1 - \alpha)(2\gamma - 1)$ , where  $\gamma$  is given by

$$\gamma = \gamma(\operatorname{Re}\lambda) = \int_0^1 (1 + t^{\operatorname{Re}\lambda})^{-1} dt$$

which is increasing function of  $\operatorname{Re}\{\lambda\}$  and  $\frac{1}{2} \leq \gamma < 1$ . The estimate is sharp in the sense that the bound can not be improved.

For real numbers  $a, b, c$  with  $c \neq 0, -1, -2, \dots$  and  $c > b > 0$ , the hypergeometric series

$$F(a, b; c; z) = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \dots$$

represents an analytic function in  $E$  [1]. The following identities are well known [1].

LEMMA 3. For real numbers  $a, b, c$  with  $c \neq 0, -1, -2, \dots$  and  $c > b > 0$ , we have

$$(1.4) \quad \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-a)}{\Gamma(c)} F(a, b; c; z),$$

$$(1.5) \quad F(a, b; c; z) = (1-z)^{-a} F(a, c-b; c; \frac{z}{z-1})$$

and

$$(1.6) \quad F(1, 1; 2; \frac{1}{2}) = 2 \ln 2.$$

## 2. Main Results

THEOREM 1. Let  $f \in \sum_{n,p}(\alpha, \delta, \mu, \lambda)$ ,  $\lambda \geq 0$ . Then

$$(2.1) \quad \operatorname{Re} \left( \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)} \right)^\mu > \frac{2(n+p)\alpha\mu + \delta\lambda}{2(n+p)\mu + \delta\lambda} \quad (0 \leq \alpha < 1, \mu > 0, z \in E),$$

where the function  $g \in \sum_p$  satisfies the condition

$$(2.2) \quad \operatorname{Re} \left( \frac{D^{n+p-1} g(z)}{D^{n+p} g(z)} \right) > \delta \quad (0 \leq \delta < 1, z \in E).$$

*Proof.* Let  $\gamma = \frac{2(n+p)\alpha\mu + \delta\lambda}{2(n+p)\mu + \delta\lambda}$  and we define the function  $q(z)$  by

$$q(z) = (1-\gamma)^{-1} \left( \left( \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)} \right)^\mu - \gamma \right).$$

Then  $q(z)$  is analytic in  $E$  and  $q(0) = 1$ . If we set  $\beta(z) = \frac{D^{n+p-1}g(z)}{D^{n+p}g(z)}$ , then by the hypothesis,  $\operatorname{Re}\{\beta(z)\} > \delta$ . Differentiating  $q(z)$  and using the identity (1.3), we have

$$\begin{aligned} (1-\lambda)\left(\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right)^\mu + \lambda\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\left(\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right)^{\mu-1} \\ = \gamma + (1-\gamma)q(z) + \frac{(1-\gamma)\lambda}{\mu(n+p)}\beta(z)zq'(z). \end{aligned}$$

Let us define the function  $\psi(r, s)$  by

$$(2.3) \quad \psi(r, s) = \gamma + (1-\gamma)r + \frac{\lambda(1-\gamma)}{\mu(n+p)}\beta(z)s.$$

Using (2.3) and the fact that  $f \in \sum_{n,p}(\alpha, \delta, \mu, \lambda)$ , we obtain

$$\{\psi(q(z), zq'(z)); z \in E\} \subset \Omega = \{\omega \in \mathbb{C}; \operatorname{Re}\{\omega\} > \alpha\}.$$

Now for all real  $r_2, s_1 \leq -\frac{1+r_2^2}{2}$ , we have

$$\begin{aligned} \operatorname{Re}\{\psi(ir_2, s_1)\} &= \gamma + \frac{\lambda(1-\gamma)s_1}{\mu(n+p)}\operatorname{Re}\{\beta(z)\} \\ &\leq \gamma - \frac{\lambda(1-\gamma)\delta(1+r_2^2)}{2\mu(n+p)} \\ &\leq \gamma - \frac{\lambda(1-\gamma)\delta}{2\mu(n+p)} = \alpha. \end{aligned}$$

Hence for each  $z \in E$ ,  $\psi(ir_2, s_1) \notin \Omega$ . Thus by Lemma 1,  $\operatorname{Re}\{q(z)\} > 0$  in  $E$  and hence

$$\operatorname{Re}\left\{\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right\}^\mu > \gamma$$

in  $E$ . This proves our theorem.

**COROLLARY 1.** Let  $f(z)$  and  $g(z)$  be in  $\sum_p$  and  $g(z)$  satisfy the condition (2.2). If  $\lambda \geq 1$  and

$$(2.4) \quad \operatorname{Re}\left\{(1-\lambda)\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} + \lambda\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\right\} > \alpha \quad (0 \leq \alpha < 1, z \in E),$$

then

$$\operatorname{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\right\} > \gamma = \frac{\alpha(2n+2p+\delta) + \delta(\lambda-1)}{2(n+p) + \lambda\delta}.$$

*Proof.* We have

$$\lambda \frac{D^{n+p}f(z)}{D^{n+p}g(z)} = \left[ (1-\lambda) \frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} + \lambda \frac{D^{n+p}f(z)}{D^{n+p}g(z)} \right] + (\lambda-1) \frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} \quad (z \in E).$$

Since  $\lambda \geq 1$ , making use of (2.4) and (2.1) (for  $\mu = 1$ ), we deduce that

$$\operatorname{Re} \left\{ \frac{D^{n+p}f(z)}{D^{n+p}g(z)} \right\} > \gamma = \frac{\alpha(2(n+p) + \delta) + \delta(\lambda - 1)}{2(n+p) + \lambda\delta}.$$

COROLLARY 2. Let  $\lambda$  be a complex number with  $\operatorname{Re}\{\lambda\} \geq 0$  ( $\lambda \neq 0$ ). If  $f \in \Sigma_p$  satisfies

$$\operatorname{Re}\{(1-\lambda)(z^p D^{n+p-1}f(z))^\mu + \lambda z^p D^{n+p}f(z)(z^p D^{n+p-1}f(z))^{\mu-1}\} > \alpha$$

for  $0 \leq \alpha < 1$  and  $\mu > 0$ , then

$$(2.5) \quad \operatorname{Re}\{z^p D^{n+p-1}f(z)\}^\mu > \frac{2\mu(n+p)\alpha + \operatorname{Re}\{\lambda\}}{2\mu(n+p) + \operatorname{Re}\{\lambda\}} \quad (z \in E).$$

Further, if  $\lambda \geq 1$  and  $f \in \Sigma_p$  satisfies

$$\operatorname{Re}\{(1-\lambda)z^p D^{n+p-1}f(z) + \lambda z^p D^{n+p}f(z)\} > \alpha \quad (z \in E),$$

then

$$(2.6) \quad \operatorname{Re}\{z^p D^{n+p}f(z)\} > \frac{(2n+2p+1)\alpha + \lambda - 1}{2(n+p) + \lambda} \quad (0 \leq \alpha < 1, z \in E).$$

*Proof.* The result (2.5) and (2.6) follows by putting  $g(z) = \frac{1}{z^p}$  in Theorem 1 and Corollary 1, respectively.

Remarks. Choosing  $n, \mu, p$  and  $\lambda$  appropriately in Corollary 2, we obtain the following results.

(i) For  $\lambda = 1$ ,  $n = 0$  and  $p = 1$  in Corollary 2, we have ;

$$(2.7) \quad \operatorname{Re}\left\{ \left(2 + \frac{zf'(z)}{f(z)}\right)(zf(z))^\mu \right\} > \alpha \quad (0 \leq \alpha < 1, z \in E)$$

implies

$$\operatorname{Re}\{zf(z)\}^\mu > \frac{2\mu\alpha + 1}{2\mu + 1} \quad (z \in E).$$

(ii) For a complex number  $\lambda$  satisfying  $\operatorname{Re}\{\lambda\} \geq 0$  ( $\lambda \neq 0$ ) and  $n = 0$ ,  $p = 1$  and  $\mu = 1$  in Corollary 2, we have ;

$$\operatorname{Re}\{(1 + \lambda)zf(z) + \lambda z^2 f'(z)\} > \alpha \quad (0 \leq \alpha < 1, z \in E)$$

implies

$$\operatorname{Re}\{zf(z)\} > \frac{2\alpha + \operatorname{Re}\{\lambda\}}{2 + \operatorname{Re}\{\lambda\}} \quad (z \in E).$$

(iii) Replacing  $f(z)$  by  $-zf'(z)$  in the result (ii), we have ;

$$-\operatorname{Re}\{(1 + 2\lambda)z^2 f'(z) + \lambda z^3 f''(z)\} > \alpha \quad (0 \leq \alpha < 1, z \in E)$$

implies

$$-\operatorname{Re}\{z^2 f'(z)\} > \frac{2\alpha + \operatorname{Re}\{\lambda\}}{2 + \operatorname{Re}\{\lambda\}} \quad (z \in E).$$

(iv) For real  $\lambda$  with  $\lambda \geq 1$  and  $n = 0$ ,  $p = 1$  and  $\mu = 1$  in Corollary 2, we have ;

$$\operatorname{Re}\{(1 + \lambda)zf(z) + \lambda z^2 f'(z)\} > \alpha \quad (0 \leq \alpha < 1, z \in E)$$

implies

$$\operatorname{Re}\{zf(z)\} > \frac{3\alpha + \lambda - 1}{2 + \lambda} \quad (z \in E).$$

**THEOREM 2.** Let  $\lambda$  be a complex number satisfying  $\operatorname{Re}\{\lambda\} > 0$ . Let  $f \in \Sigma_p$  satisfy the condition

$$(2.8) \quad \operatorname{Re}\{(1 - \lambda)(z^p D^{n+p-1} f(z))^\mu + \lambda z^p D^{n+p} f(z)(z^p D^{n+p-1} f(z))^{\mu-1}\} > \alpha$$

for some  $\alpha(0 \leq \alpha < 1)$  and  $\mu > 0$ . Then

$$(2.9) \quad \operatorname{Re}\{z^p D^{n+p-1} f(z)\}^\mu > \alpha + (1 - \alpha)(2\rho - 1),$$

where

$$\rho = \frac{1}{2}F(1, 1; 1 + \frac{\mu(n+p)}{\operatorname{Re}\{\lambda\}}; \frac{1}{2}).$$

*Proof.* Let  $q(z) = (z^p D^{n+p-1} f(z))^\mu$ . Then  $q(z)$  is analytic in  $E$  with  $q(0) = 1$ . Differentiating  $q(z)$  and using the identity (1.3), we get

$$\begin{aligned} & (1 - \lambda)(z^p D^{n+p-1} f(z))^\mu + \lambda z^p D^{n+p} f(z)(z^p D^{n+p-1} f(z))^{\mu-1} \\ &= q(z) + \frac{\lambda z q'(z)}{\mu(n+p)}, \end{aligned}$$

so that by the hypothesis (2.8), we have

$$\operatorname{Re}\left\{q(z) + \frac{\lambda z q'(z)}{\mu(n+p)}\right\} > \alpha \quad (z \in E).$$

In view of Lemma 2, this implies that

$$\operatorname{Re}\{q(z)\} > \alpha + (1 - \alpha)(2\rho - 1),$$

where

$$\rho = \rho(\operatorname{Re}\{\lambda\}) = \int_0^1 (1 + t^{\operatorname{Re}\frac{\lambda}{\mu(n+p)}})^{-1} dt.$$

Setting  $\operatorname{Re}\{\lambda\} = \lambda_1 > 0$ , we have

$$\begin{aligned} \rho &= \int_0^1 (1 + t^{\frac{\lambda_1}{\mu(n+p)}})^{-1} dt \\ &= \frac{\mu(n+p)}{\lambda_1} \int_0^1 u^{\frac{\mu(n+p)}{\lambda_1}-1} (1+u)^{-1} du. \end{aligned}$$

Using (1.4) and (1.5), we get

$$\begin{aligned} \rho &= F\left(1, \frac{\mu(n+p)}{\lambda_1}; 1 + \frac{\mu(n+p)}{\lambda_1}; -1\right) \\ &= \frac{1}{2} F\left(1, 1; 1 + \frac{\mu(n+p)}{\lambda_1}; \frac{1}{2}\right). \end{aligned}$$

**COROLLARY 3.** Let  $\lambda$  be a real number satisfying  $\lambda \geq 1$ . If  $f \in \sum_p$  satisfies

$$\operatorname{Re}\{(1 - \lambda)z^p D^{n+p-1} f(z) + \lambda z^p D^{n+p} f(z)\} > \alpha \quad (z \in E)$$

for  $\alpha(0 \leq \alpha < 1)$ , then

$$\operatorname{Re}(z^p D^{n+p} f(z)) > \alpha + (1 - \alpha)(2\rho' - 1)(1 - \lambda^{-1}) \quad (z \in E),$$



where

$$\rho' = \frac{1}{2}F(1, 1; 1 + \frac{n+p}{\lambda}; \frac{1}{2}).$$

*Proof.* The result follows by using the identity

$$\lambda z^p D^{n+p} f(z) = [(1 - \lambda)z^p D^{n+p-1} f(z) + \lambda z^p D^{n+p} f(z)] + (\lambda - 1)z^p D^{n+p-1} f(z).$$

Remark. We note that if  $n = 0$ ,  $p = 1$ ,  $\mu = \lambda > 0$  in Corollary 2, that is,

$$(2.10) \quad \operatorname{Re}\{(1 - \lambda)(zf(z))^\lambda + \lambda z^2 f'(z)(zf(z))^{\lambda-1}\} > \alpha \quad (0 \leq \alpha < 1, z \in E),$$

then (2.5) implies that

$$(2.11) \quad \operatorname{Re}(zf(z))^\lambda > \frac{2\alpha + 1}{3} \quad (z \in E).$$

Whereas, if  $f \in \sum_1$  satisfies the condition (2.10), then by using Theorem 2,

$$\operatorname{Re}(zf(z))^\lambda > 2(1 - \ln 2)\alpha + (2\ln 2 - 1) \quad (z \in E),$$

which is better than (2.11).

Similarly, if  $n = p = 1$ ,  $\lambda = 2$  in (2.6) and

$$\operatorname{Re}\{2zD^2 f(z) - zDf(z)\} > \alpha \quad (z \in E),$$

then we have

$$\operatorname{Re}(zD^2 f(z)) > \frac{5\alpha + 1}{6} \quad (z \in E).$$

Whereas, by using (1.6), Corollary 3 implies that

$$\operatorname{Re}(zD^2 f(z)) > \frac{1}{2}[(3 - 2\ln 2) + (2\ln 2 - 1)] > \frac{5\alpha + 1}{6}.$$

We observe that if  $\lambda$  is a real number satisfying  $\lambda > 0$  and

$$h(z) = \frac{D^{n+p} f(z)}{D^{n+p} g(z)} + \left(\frac{1}{\lambda} - 1\right) \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)} \quad (z \in E),$$

then from Theorem 1 (for  $\mu = 1$ ), we have ;

$$(2.12) \quad \operatorname{Re} h(z) > \frac{\alpha}{\lambda} \text{ implies } \operatorname{Re} \left( \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)} \right) > \frac{2(n+p)\alpha + \lambda\delta}{2(n+p) + \lambda\delta},$$

whenever

$$\operatorname{Re} \left( \frac{D^{n+p-1} g(z)}{D^{n+p} g(z)} \right) > \delta \quad (0 \leq \delta < 1, z \in E).$$

Let  $\lambda \rightarrow +\infty$ , then from (2.12) we have ;

$$\operatorname{Re} h(z) \geq 0 \text{ in } E \text{ implies } \operatorname{Re} \left( \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)} \right) \geq 1 \text{ in } E,$$

whenever

$$\operatorname{Re} \left( \frac{D^{n+p-1} g(z)}{D^{n+p} g(z)} \right) > \delta \quad (0 \leq \delta < 1, z \in E).$$

In the following theorem, we shall extend the above results as follows.

**THEOREM 3.** Suppose  $f(z)$  and  $g(z)$  are in  $\Sigma_p$  and  $g(z)$  satisfies the condition (2.2). If

$$(2.13) \quad \operatorname{Re} \left\{ \frac{D^{n+p} f(z)}{D^{n+p} g(z)} - \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)} \right\} > -\frac{(1-\alpha)\delta}{2(n+p)} \quad (z \in E)$$

for some  $\alpha(0 \leq \alpha < 1)$ , then

$$(2.14) \quad \operatorname{Re} \left\{ \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)} \right\} > \alpha \quad (z \in E)$$

and

$$(2.15) \quad \operatorname{Re} \left\{ \frac{D^{n+p} f(z)}{D^{n+p} g(z)} \right\} > \frac{(2n+2p+\delta)\alpha - \delta}{2(n+p)} \quad (z \in E).$$

*Proof.* Let

$$q(z) = (1-\alpha)^{-1} \left[ \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)} - \alpha \right].$$

Then  $q(z)$  is analytic in  $E$  with  $q(0) = 1$ . Setting

$$\beta(z) = \frac{D^{n+p-1} g(z)}{D^{n+p} g(z)} \quad (z \in E),$$

we observe that by hypothesis,  $\operatorname{Re}\{\beta(z)\} > \delta$  in  $E$ . A simple computation shows that

$$\begin{aligned} \frac{(1-\alpha)zq'(z)\beta(z)}{n+p} &= \frac{D^{n+p}f(z)}{D^{n+p}g(z)} - \frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} \\ &= \psi(q(z), zq'(z)), \end{aligned}$$

where

$$\psi(r, s) = \frac{(1-\alpha)\beta(z)s}{n+p}.$$

Using the hypothesis (2.13), we get

$$\{\psi(q(z), zq'(z)); z \in E\} \subset \Omega = \{\omega \in \mathbb{C} : \operatorname{Re} \omega > -\frac{(1-\alpha)\delta}{2(n+p)}\}.$$

Now, for all real  $r_2, s_1 \leq -\frac{(1+r_2^2)}{2}$ , we have

$$\begin{aligned} \operatorname{Re}\{\psi(ir_2, s_1)\} &= \frac{s_1(1-\alpha)\operatorname{Re}\{\beta(z)\}}{n+p} \\ &\leq -\frac{(1-\alpha)\delta(1+r_2^2)}{2(n+p)} \\ &\leq -\frac{(1-\alpha)\delta}{2(n+p)}. \end{aligned}$$

This shows that  $\psi(ir_2, s_1) \notin \Omega$  for each  $z \in E$ . Hence by Lemma 1,  $\operatorname{Re}\{q(z)\} > 0$  in  $E$ . This proves (2.14). The proof of (2.15) follows by using (2.13) and (2.14) in the identity

$$\operatorname{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\right\} = \operatorname{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p}g(z)} - \frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right\} + \operatorname{Re}\left\{\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right\}.$$

This completes the proof of Theorem 3.

Remarks. (i) For  $n = 0$ ,  $p = 1$  and  $g(z) = \frac{1}{z}$  in Theorem 3, we have ;

$$\operatorname{Re}(zf(z) + z^2f'(z)) > -\frac{(1-\alpha)}{2} \quad (z \in E)$$

implies

$$\operatorname{Re}(zf(z)) > \alpha \quad (z \in E)$$

and

$$\operatorname{Re}(2zf(z) + z^2f'(z)) > \frac{3\alpha - 1}{2} \quad (z \in E).$$

(ii) For  $n = p = 1$ ,  $g(z) = \frac{1}{z}$  in Theorem 3, we have ;

$$\operatorname{Re}(zD^2f(z) - zDf(z)) > -\frac{1-\alpha}{4} \quad (z \in E)$$

implies

$$\operatorname{Re}(zDf(z)) > \alpha \quad (z \in E)$$

and

$$\operatorname{Re}(zD^2f(z)) > \frac{5\alpha - 1}{4} \quad (z \in E).$$

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